Chapter 12: Exponentials & Logarithms

12.1 Composite functions
Inverse functions

12.2 Exponential Functions

12.3 Logarithmic functions

12.4 Properties of logarithms

12.5 Natural Logarithms

12.6 Solving log + Exponential Equations

12.7 Applications

* Logarithms are a key component of Math 70, together with word problems and learning the graphing calculator.
The next exam will be only chapter 12.

12.1 - 1st

objectives

1) Find the composition of two functions and fully simplify
2) Identify functions which are composed to get a given function.
3) Observe that some functions "undo" each other when composed, while others do not.

Math 70 12.1 Algebra of Functions

Overview of Chapter 12	
12.1	Function Composition
	Inverse Functions [Functions that "un-do" each other when composed]
12.2	Exponential Functions
12.3	Logarithmic Functions [Inverse functions (12.1) of Exponential Functions (12.2)]
12.4	Properties of Logarithms [Unexpected traits of Logarithmic functions (12.3)]
12.5	Natural logs, and Change of Base [Special logarithmic functions(12.3) and GC]
12.6 & 12.7	Solving Exponential and Logarithmic Equations and Applications
*This chapter builds one section on the next, layering complicated concepts.	

Objectives

1) Find a new function which is composition $(f \circ g)(x)$ of two given functions.
2) Review 5.9: the sum $(f+g)(x)$, difference $(f-g)(x)$, product $(f \cdot g)(x)$, quotient $(f/g)(x)$
3) Recognize function notation and notation for the names of these functions.
4) Practice negative and positive exponents on common bases, in preparation for 12.2.

Practice and Examples

1) Given $f(x)=3x^2+4x+1$ and $g(x)=2x-5$, find:
   a) $(f+g)(x)$
   b) $(f-g)(x)$
   c) $(f \cdot g)(x)$
   d) $(f/g)(x)$
   e) $(g \circ f)(x)$
   f) $(f \circ g)(x)$

2) Given $\begin{cases} f(-1)=4 & g(-1)=-4 \\ f(0)=5 & g(0)=-3 \\ f(2)=7 & g(2)=-1 \\ f(7)=1 & g(7)=9 \end{cases}$, find
   a) $(f+g)(2)$
   b) $(f-g)(0)$
   c) $(f \cdot g)(7)$
   d) $(f \cdot g)(0)$
   e) $(f/g)(0)$
   f) $(g/f)(0)$
   g) $(g \circ f)(2)$
   h) $(f \circ g)(2)$

3) On interstate trips, a driver averages 54 mph. The distance d in miles traveled in t hours is given by $d(t) = 54t$. Because the driver averages 25 miles per gallon, the number of gallons g used is given by $g(d) = d/25$. The cost per gallon is $2.95, so the total fuel cost is given by $c(g) = 2.95g$.

   a) Write a function describing the number of gallons used in t hours of travel.
   b) Write a function describing the total fuel cost in t hours of travel.
   c) Determine the total fuel cost of a 12-hour trip.

4) An oil tanker runs aground and springs a leak. The oil spreads out in a semicircular pattern from the shoreline. The distance r (in feet) from the tanker to the edge of the oil spill at time t (in minutes) is given by the function $r(t) = 20t$.

   a) If the area of the semicircle is given by $A(x) = \frac{1}{2}\pi x^2$, where x is the radius, write a function for the area covered by the oil at time t.
   b) What is the area of the oil spill after 5 minutes?

5) Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find:

   a) $(g \circ f)(x)$
   b) $(f \circ g)(x)$

Recall: Function Notation $f(x)$ means "f of x" where f is the name of the function and (x) tells the variable used in the function.

In 9.1 we will write $(f+g)(x)$, spoken "f plus g of x", which means the name of the function is "f plus g" and the variable being used is x.

CAUTION: "of x" does not mean "multiply by x"

① Given $f(x)=3x^2+4x+1$ and $g(x)=2x-5$, find

a) $(\underbrace{f+g}_{\text{name of new function}})\underbrace{(x)}_{\text{function notation "of x" indicates variable used.}}$.

f+g is the name of the new function

(f+g)(x) is pronounced "f plus g, of x"

Method: Add f(x) and g(x).
$(f+g)(x) = f(x)+g(x)$.

$= f(x)+g(x)$

$= (3x^2+4x+1)+(2x-5)$ — subst expressions for f(x) & g(x).

$= \boxed{3x^2+6x-4}$ — combine like terms = add.

b) $(f-g)(x)$

"f−g" is the name of the new function — "f minus g, of x"

$= f(x)-g(x)$

$= (3x^2+4x+1)-(2x-5)$ — substitute expressions
* must use ( )

$= 3x^2+4x+1-2x+5$ — distribute negative

$= \boxed{3x^2+2x+6}$ — combine like terms

c) $(f\cdot g)(x)$

Handwriting alert! $f\cdot g$ is different from $f\circ g$

small dot = multiply

loop = compose

M70 M-G

$= f(x) \cdot g(x)$

$= (3x^2+4x+1)(2x-5)$

"f times g, of x"
or simply "fg of x"
substitute
* must use ( )

$= 3x^2(2x-5) + 4x(2x-5) + 1(2x-5)$

distribute each term of first

$= 6x^3 - 15x^2 + 8x^2 - 20x + 2x - 5$

$= 6x^3 \quad x^2 - 18x - 5$

combine

d) $(f/g)(x)$

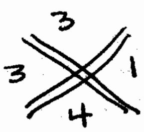
"f divide by g, of x"

$= \dfrac{f(x)}{g(x)}$

$= \dfrac{3x^2+4x+1}{2x-5}$

factor and cancel to simplify fraction, if possible

$= \boxed{\dfrac{(3x+1)(x+1)}{(2x-5)}}$



$\underbrace{3x^2+3x}+\underbrace{x+1}$

$= 3x(x+1) + 1(x+1)$

$= (x+1)(3x+1)$

Putting one function value inside another is called <u>function composition</u>.
This is the most important skill in 9.1
because we need it to

- un-do functions (inverse functions)
- un-do exponential functions specifically (logarithms).

$f(g(x))$ is also called $(f \circ g)(x)$
or "f composed on g of x".

e) $(g \circ f)(x)$ "g composed on f, of x"

$= g(f(x))$ keep the order of the functions

$= 2(\ \ ) - 5$

replace all x's in g(x) by f(x)

$= 2(f(x)) - 5$ subst for f(x)

$= 2(3x^2 + 4x + 1) - 5$ simplify.

$= 6x^2 + 8x + 2 - 5$ distribute 2

$= \boxed{6x^2 + 8x - 3}$ combine

f) $(f \circ g)(x)$ "f composed on g, of x"

$= f(g(x))$ keep the order of the functions

$= 3(\ \ )^2 + 4(\ \ ) + 1$

replace all x's in f(x) by g(x)

$= 3(g(x))^2 + 4(g(x)) + 1$ substitute for g(x)

$= 3(2x-5)^2 + 4(2x-5) + 1$ simplify using order of operations
- exponents
- then multiply
- then add/subtract L→R.

$= 3\underbrace{(2x-5)(2x-5)}_{\text{exponent = FOIL}} + 4(2x-5) + 1$

$= 3(4x^2 - 20x + 25) + 4(2x-5) + 1$ distribute

$= 12x^2 - 60x + 75 + 8x - 20 + 1$ combine

$= \boxed{12x^2 - 52x + 56}$

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(2) If we are given that

$f(-1)=4$ $g(-1)=-4$
$f(0)=5$ $g(0)=-3$
$f(2)=7$ $g(2)=-1$
$f(7)=1$ $g(7)=9$

Find

a) $(f+g)(2)$ "f plus g of 2"

$= f(2)+g(2)$ method

$= 7+(-1)$ substitute given values

$= \boxed{6}$

If we graph $y_1 = f(x)$
$y_2 = g(x)$
$y_3 = (f+g)(x)$

and look at a table of values, we will see that, at a particular value of x, $y_3 = y_2 + y_1$ is the sum of the y-values

b) $(f-g)(0)$ "f minus g of zero"

$= f(0)-g(0)$ method

$= 5-(-3)$ substitute

$= 5+3$

$= \boxed{8}$

c) $(f\cdot g)(7)$ "f times g of 7"

$= f(7)\cdot g(7)$ method

$= 1\cdot 9$ substitute

$= \boxed{9}$

Math 70

d) $(f \cdot g)(0)$ "f times g of zero"

$= f(0) \cdot g(0)$ method

$= (5)(-3)$ substitute

$= \boxed{-15}$

e) $(f/g)(0)$ "f divide by g of zero"

$= f(0) / g(0)$ method

$= 5/(-3)$ substitute

$= \boxed{-\frac{5}{3}}$

NOTICE: Not multiply by zero.
Yes: Evaluate at zero.

f) $(g/f)(0)$ "g divide by f of zero"

$= g(0) / f(0)$

$= -3/5$

$= \boxed{\frac{-3}{5}}$

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g) $(g \circ f)(2)$

Find $g(f(2))$

step 1: find $f(2) = 7$

step 2: put result into g

find $g(7) = 9$

answer $\boxed{9}$

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h) $(f \circ g)(2)$ Find $f(g(2))$

work from the inside out

step 1: find $g(2) = -1$

step 2: put this result (-1) into f

find $f(-1) = 4$

answer $= \boxed{4}$

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$(f \circ g)(x) = f(g(x))$ is not the same as $(g \circ f)(x) = g(f(x))$ as we saw in our previous work:

1 e) $(g \circ f)(x) = 6x^2 + 8x - 3$

f) $(f \circ g)(x) = 12x^2 - 52x + 56$

2 g) $(g \circ f)(x) = 9$

h) $(f \circ g)(x) = 4$

3 On interstate trips, a driver averages 54 mph. The distance d in miles traveled in t hours is given by $d(t) = 54t$. Because the driver averages 25 miles per gallon, the number of gallons g used is given by $g(d) = d/25$. The cost per gallon is \$2.95, so the total fuel cost is given by $c(g) = 2.95g$.

a) Write a function describing the number of gallons used in t hours of travel.
b) Write a function describing the total fuel cost in t hours of travel.
c) Determine the total fuel cost of a 12-hour trip.

a) $g(d(t)) = \frac{54t}{25} = \boxed{2.16t}$

b) $c(g(d(t))) = 2.95(2.16t) = \boxed{6.372t}$

c) $c(g(d(t))) = (6.372)(12) = \$76.464$
$\approx \boxed{\$76.46}$

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4 An oil tanker runs aground and springs a leak. The oil spreads out in a semicircular pattern from the shoreline. The distance r (in feet) from the tanker to the edge of the oil spill at time t (in minutes) is given by the function $r(t) = 20t$.

a) If the area of the semicircle is given by $A(x) = \frac{1}{2}\pi x^2$, where x is the radius, write a function for the area covered by the oil at time t.
b) What is the area of the oil spill after 5 minutes?

a) $A(r(t)) = \frac{1}{2}\pi(20t)^2$
$= \frac{1}{2}\pi \cdot 400t^2$
$= \boxed{200\pi t^2}$

b) $A(r(5)) = 200\pi(5)^2$
$= 200 \cdot 25 \cdot \pi$
$= \boxed{5000\pi \text{ ft}^2}$
$\approx \boxed{15{,}707.96 \text{ ft}^2}$

This is function composition also.

$A(r(t))$ is the method (replace x in $A(x)$ by expression $r(t) = 20t$).

$(A \circ r)(t)$ means "A composed on r of t" and this is the name of the function

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5) Given $f(x) = 2x+3$
$g(x) = \frac{1}{2}(x-3)$

a) Find $(g \circ f)(x) = g(f(x))$
$= \frac{1}{2}[(2x+3)-3]$
$= \frac{1}{2}[2x]$
$\boxed{(g \circ f)(x) = x}$

b) Find $(f \circ g)(x) = f(g(x))$
$= 2[\frac{1}{2}(x-3)] + 3$
$= x - 3 + 3$
$\boxed{(f \circ g)(x) = x}$

What does this mean, that $f(g(x)) = x$?
Pick a random number —
$x = 7$.

Find $(f \circ g)(7) = f(g(7)) = f(\underbrace{\tfrac{1}{2}(4)}_{\frac{1}{2}(7-3)}) = f(2) = 2(2)+3 = \boxed{7}$

$x=7$ in   f and g un-do each other $x = 7$ out

Similarly, $g(f(x)) = x$ for a random value of x —
$x = -23$.

Find $(g \circ f)(-23) = g(f(-23)) = \boxed{-23}$

$x = -23$ in $x = -23$ out

So $f(x) = 2x+3$ and $g(x) = \frac{1}{2}(x-3)$ have a special relationship to each other because when we compose them, they un-do each other.
This special relationship is the focus of section 9.2.

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We will be using exponents a lot.

The worksheet on the next page is a review of exponents using common bases and common whole-number exponents.

Key

Math 70 Practice with Exponents

Complete the table by raising each base to each exponent. Some example entries are provided.

Exponents across → Bases down ↓	-3	-2	-1	0	2	3
0	$\frac{1}{0^3} = 0^{-3} = \boxed{\text{undefined}}$	$\frac{1}{0^2} = \boxed{\text{undefined}}$	$\frac{1}{0^{-1}} = \boxed{\text{undefined}}$	0^0 indeterminate	0	0
1	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	1	1	1
2	$2^{-3} = \frac{1}{2^3} = \boxed{\frac{1}{8}}$	$2^{-2} = \boxed{\frac{1}{4}}$	$2^{-1} = \frac{1}{2^1} = \boxed{\frac{1}{2}}$	1	4	8
3	$3^{-3} = \frac{1}{3^3} = \boxed{\frac{1}{27}}$	$\boxed{\frac{1}{9}}$	$3^{-1} = \boxed{\frac{1}{3}}$	$3^0 = 1$	9	27
4	$4^{-3} = \frac{1}{4^3} = \boxed{\frac{1}{64}}$	$\boxed{\frac{1}{16}}$	$\boxed{\frac{1}{4}}$	1	16	64
5	$5^{-3} = \frac{1}{5^3} = \boxed{\frac{1}{125}}$	$\boxed{\frac{1}{25}}$	$\boxed{\frac{1}{5}}$	1	$5^2 = 25$	125
6	$6^{-3} = \frac{1}{6^3} = \boxed{\frac{1}{216}}$	$\boxed{\frac{1}{36}}$	$\boxed{\frac{1}{6}}$	1	36	$6^3 = 216$
-1	$(-1)^{-3} = \frac{1}{(-1)^3} = \boxed{-1}$	$(-1)^{-2} = \left(\frac{1}{-1}\right)^2 = \boxed{1}$	$(-1)^{-1} = \frac{1}{-1} = \boxed{-1}$	1	$(-1)^2 = 1$	$(-1)^3 = -1$
-2	$(-2)^{-3} = \frac{1}{(-2)^3} = \boxed{-\frac{1}{8}}$	$(-2)^{-2} = \left(\frac{1}{-2}\right)^2 = \boxed{\frac{1}{4}}$	$(-2)^{-1} = \frac{1}{-2} = \boxed{-\frac{1}{2}}$	1	$(-2)^2 = 4$	$(-2)^3 = -8$
-3	$(-3)^{-3} = \frac{1}{(-3)^3} = \boxed{-\frac{1}{27}}$	$\boxed{\frac{1}{9}}$	$(-3)^{-1} = \frac{1}{-3} = \boxed{-\frac{1}{3}}$	1	$(-3)^2 = 9$	-27
-4	$(-4)^{-3} = \frac{1}{(-4)^3} = \boxed{-\frac{1}{64}}$	$\boxed{\frac{1}{16}}$	$\boxed{-\frac{1}{4}}$	1	16	-64
$\frac{1}{2}$	$\left(\frac{1}{2}\right)^{-3} = \left(\frac{2}{1}\right)^3 = \boxed{8}$	$\left(\frac{1}{2}\right)^{-2} = (2)^2 = \boxed{4}$	$\left(\frac{1}{2}\right)^{-1} = \boxed{2}$	1	$\frac{1}{4}$	$-\frac{1}{8}$
$\frac{1}{3}$	$\left(\frac{1}{3}\right)^{-3} = (3)^3 = \boxed{27}$	$\boxed{9}$	$\boxed{3}$	1	$\frac{1}{9}$	$-\frac{1}{27}$
$\frac{1}{4}$	$\boxed{64}$	$\boxed{16}$	$\boxed{4}$	1	$\frac{1}{16}$	$-\frac{1}{64}$
$\frac{1}{5}$	$\boxed{125}$	$\boxed{25}$	$\boxed{5}$	1	$\frac{1}{25}$	$-\frac{1}{125}$
$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^{-3} = (-2)^3 = \boxed{-8}$	$\left(-\frac{1}{2}\right)^{-2} = (-2)^2 = \boxed{4}$	$\left(-\frac{1}{2}\right)^{-1} = (-2)^{+1} = \boxed{-2}$	1	$\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$
$-\frac{1}{3}$	$\left(-\frac{1}{3}\right)^{-3} = (-3)^3 = \boxed{-27}$	$\boxed{9}$	$\boxed{-3}$	1	$\frac{1}{9}$	$-\frac{1}{27}$
$-\frac{1}{4}$	$\left(-\frac{1}{4}\right)^{-3} = \left(-\frac{4}{1}\right)^3 = \boxed{-64}$	$\boxed{16}$	$\boxed{-4}$	1	$\frac{1}{16}$	$-\frac{1}{64}$
$-\frac{1}{5}$	$\boxed{-125}$	$\left(-\frac{1}{5}\right)^{-2} = \left(-\frac{5}{1}\right)^2 = \boxed{25}$	$\boxed{-5}$	1	$\frac{1}{25}$	$-\frac{1}{125}$